



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 9

1 Matrices and matrix operations

Exercise 1

- (a) Write down the size of each of the following matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 3 \\ x & -5 \\ 0 & 2 \end{pmatrix},$$

$$\mathbf{C} = (0 \quad -3 \quad 8).$$

- (b) Using the matrices in part (a), write down the following elements if they exist:

$$a_{21}, a_{22}, b_{21}, b_{32}, c_{13}, c_{31}.$$

Exercise 2

Calculate all the possible sums of two different matrices from the following list:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 3 \\ x & -5 \\ 0 & 2 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -1 & 7 \\ 4 & -3 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 5 & 0 & 3 \\ -2 & y & 6 \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} 6 & 2 \\ 0.5 & 3 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 6 & -5 \\ 1 & 2 \\ 5 & 4 \end{pmatrix}.$$

Exercise 3

Use the matrices from Exercise 2.

- (a) Where possible, calculate the following matrices.

(i) $\mathbf{A} - \mathbf{C}$ (ii) $\mathbf{F} - \mathbf{B}$

(iii) $\mathbf{A} - \mathbf{E} - \mathbf{C}$ (iv) $\mathbf{E} - \mathbf{F}$

(v) $\mathbf{D} + \mathbf{D}$

- (b) Calculate the following products of scalars and matrices.

(i) $0.5\mathbf{A}$ (ii) $2\mathbf{D}$ (iii) $3\mathbf{E}$

- (c) What do you notice about your answers to (a)(v) and (b)(ii)?

Exercise 4

Simplify each of the following matrices by writing it as a scalar multiple of a matrix with integer coefficients. Make the magnitude of the integer coefficients as small as possible.

(a) $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix}$ (b) $(3x \quad -5x \quad -2x)$

(c) $\begin{pmatrix} 3.2 & 5.4 \\ -2.4 & 1.2 \end{pmatrix}$

Exercise 5

Let

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 4 \\ -5 & 6 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -4 & 7 \\ 1 & 3 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 5 & -2 \\ 4 & 7 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 0 & 2 & -5 \\ 3 & 1 & 6 \end{pmatrix}.$$

Where possible, evaluate the following matrices.

(a) $\mathbf{A} + 2\mathbf{D}$ (b) $2\mathbf{B} + 3\mathbf{C}$

(c) $2\mathbf{C} - 3\mathbf{B}$ (d) $\mathbf{D} - 2\mathbf{A}$

(e) $3\mathbf{B} - \mathbf{D}$

Exercise 6

- (a) Write down the size of each of the following matrices:

$$\mathbf{M} = (2 \quad 0 \quad -3), \quad \mathbf{N} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}.$$

- (b) Using the matrices in part (a), show that the matrix products $\mathbf{MN}$ and $\mathbf{NM}$ are defined, and determine their sizes. Calculate these matrices.

Exercise 7

Let

$$\mathbf{P} = \begin{pmatrix} 0 & 2 \\ 4 & -1 \\ -3 & 5 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & -4 & 0 \\ 2 & 5 & 3 \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} -2 & 6 \\ 4 & 5 \\ -1 & 2 \end{pmatrix}.$$

By considering the sizes of the matrices above, determine whether each of the following products is defined and if so, what the size of the product matrix will be. Then, where possible, calculate each product.

- (a) $\mathbf{PQ}$ (b) $\mathbf{SR}$ (c) $\mathbf{RS}$
 (d) $\mathbf{QS}$ (e) $\mathbf{SQ}$

Exercise 8

- (a) Use the matrices in Exercise 7 to calculate the following products.
 (i) $\mathbf{PR}$ (ii) $2\mathbf{P}(3\mathbf{R})$
 (iii) $6\mathbf{PR}$ (iv) $(-2\mathbf{P})(3\mathbf{R})$
 (b) What do you notice about your solutions?

Exercise 9

By calculating $(\mathbf{A} + \mathbf{B})\mathbf{C}$ and $\mathbf{AC} + \mathbf{BC}$ in each example below, confirm the property that $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

(a) $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 8 & -7 \end{pmatrix},$

$$\mathbf{C} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}.$$

(b) $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$

$$\mathbf{C} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Exercise 10

- (a) By considering the size of each of the following matrices, decide which ones can be squared:

$$\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 5 & 4 \end{pmatrix}, \quad \mathbf{B} = (-4),$$

$$\mathbf{C} = \begin{pmatrix} -1 & 7 \\ 4 & 5 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 3 \\ -2 & 5 & 6 \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} 6 & -5 \\ 1 & 2 \\ 5 & 4 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 1 & -2 & 6 \\ 3 & 0 & 4 \\ 2 & 5 & 8 \end{pmatrix}.$$

- (b) Where possible, calculate the squares of the matrices in part (a).

Exercise 11

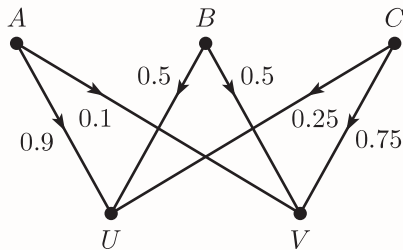
Let $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}.$

- (a) Calculate the following powers of $\mathbf{M}$.
 (i) $\mathbf{M}^2$ (ii) $\mathbf{M}^3$ (iii) $\mathbf{M}^4$
 (b) Use your answers to part (a) to calculate $\mathbf{M}^6$.

2 Matrices and networks

Exercise 12

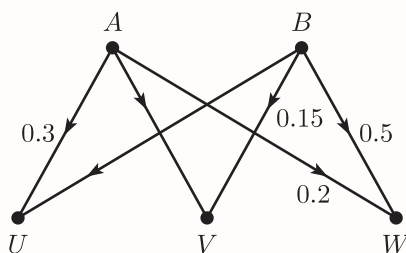
The following network diagram represents a pipe system in which water flows from input nodes A , B , and C to output nodes U and V .



- How much water is output at node V if 5 litres of water are input at node A ?
- How much water is output at nodes U and V if 2 litres of water are input at each of nodes A and C , and 1 litre is input at node B ?
- How much water is output at nodes U and V if x , y and z litres of water are input at nodes A , B and C , respectively?
- Write down the vector that represents the input in part (b).
- Write down the matrix that represents this network.

Exercise 13

The following network diagram represents a pipe system in which water flows from input nodes A and B to output nodes U , V and W .

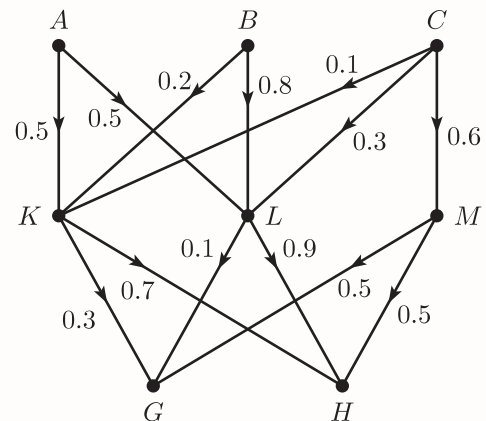


- What are the missing numbers for the arcs from A to V and from B to U ?
- Write down the matrix that represents this network.
- Suppose that x litres of water are input at node A and y litres at node B , and that the outputs are u , v and w litres at nodes U , V and W , respectively. Write down a matrix equation that connects the corresponding input and output vectors.

Exercise 14

The network below is the combination of two smaller networks, one with input nodes A , B and C and output nodes K , L and M , and the other with input nodes K , L and M and output nodes G and H .

Some pairs of input and output nodes are not connected with an arc. Each such missing arc is equivalent to an arc with label 0; in other words, there is no flow from the input node to the output node.



- Write down the matrix equation that gives the output vector $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ at nodes K , L and M when the input vector at nodes A , B and C is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
- Write down the matrix equation that gives the output vector $\begin{pmatrix} g \\ h \end{pmatrix}$ at nodes G and H when the input vector at nodes K , L and M is $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$.

- (c) Use your answers to parts (a) and (b) to write down the matrix equation for the combined network when the input vector at nodes A , B and C is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.
- (d) Draw a simplified network with input nodes A , B and C and output nodes G and H that is equivalent to the given network.
- (e) Suppose that the original network represents the principal transport routes from towns A , B and C to airports G and H . The labels on the arcs give the proportions of travellers choosing each transport link from each respective location. Assume that all travellers use these main routes to the airports. If the numbers of these travellers leaving from towns A , B and C each month are 120, 50 and 100, how many of these people arrive at airport G and how many arrive at airport H ?

3 The inverse of a matrix

Exercise 15

Let

$$\mathbf{P} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 6 & -1 \\ -4 & 2 \\ 0 & 3 \end{pmatrix},$$

$$\mathbf{R} = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 1 & 5 \end{pmatrix} \text{ and } \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- (a) Which of the following matrix products can be formed?

PI, IP, QI, IQ, RI, IR.

- (b) Where possible, find each product.

Exercise 16

Let

$$\mathbf{L} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 3 & -5 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 4 & -2 \\ 3 & 5 \\ -1 & 0 \end{pmatrix},$$

$$\mathbf{N} = \begin{pmatrix} 0 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix} \text{ and } \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- (a) Which of the following matrix products can be formed?

LI, IL, MI, IM, NI, IN.

- (b) Where possible, find each product.

Exercise 17

For each of the following pairs of matrices $\mathbf{P}$ and $\mathbf{Q}$, check that $\mathbf{Q}$ is the inverse of $\mathbf{P}$ by finding the products $\mathbf{PQ}$ and $\mathbf{QP}$.

(a) $\mathbf{P} = \begin{pmatrix} -6 & 2 \\ -2 & 1 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & 3 \end{pmatrix}.$

(b) $\mathbf{P} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & -4 \\ 0 & 5 & 1 \end{pmatrix},$

$$\mathbf{Q} = \begin{pmatrix} -19 & 10 & 2 \\ 2 & -1 & 0 \\ -10 & 5 & 1 \end{pmatrix}.$$

Exercise 18

For each of the following matrices, find its inverse or show that the inverse does not exist.

(a) $\mathbf{A} = \begin{pmatrix} 7 & 2 \\ 6 & 2 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$

(c) $\mathbf{C} = \begin{pmatrix} \frac{1}{3} & \frac{1}{5} \\ 5 & -\frac{1}{2} \end{pmatrix}$

Exercise 19

This exercise concerns the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

- (a) Find the determinant of the matrix $\mathbf{A}$.
- (b) For which values of a and b does $\mathbf{A}$ have an inverse? For these values, write down the inverse matrix.

Exercise 20

Let $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.

- (a) What is the determinant of $\mathbf{A}$?
- (b) Calculate the matrix products $\mathbf{AB}$ and $\mathbf{BA}$. What conclusion can you draw?

Exercise 21

Use the computer algebra system to determine which of the following matrices are invertible. For those matrices that are invertible, find the inverse.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -2 & 1 & 2 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 2 & 0 & -4 \\ 4 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 3 & -4 \\ 0 & -3 & 1 & 2 \\ 1 & -2 & 0 & -2 \\ 0 & 1 & -1 & 3 \end{pmatrix}.$$

4 Simultaneous linear equations and matrices

Exercise 22

For each of the following pairs of simultaneous linear equations, rewrite the equations in matrix form and use matrix methods to solve the equations.

- (a) $4x + y = 2$
 $3x + y = 5$
- (b) $-2x + 5y = 8$
 $3x + 7y = -2$
- (c) $\frac{1}{3}x + 3y = -8$
 $\frac{1}{2}x - \frac{3}{2}y = 6$

Exercise 23

Determine whether the following pairs of simultaneous linear equations have a unique solution. (You are not asked to find the solution.)

- (a) $7x + 3y = -1$
 $-3x + y = 6$
- (b) $-4x + 6y + 6 = 0$
 $9y = 6x - 5$

Exercise 24

- (a) Write each of the following systems of simultaneous linear equations in matrix form.
- (i) $3x + 2y + z = 4$
 $-5x + 2y + 2z = 15$
 $2x - 3y + 3z = 1$
 - (ii) $3x - 3y + 2z = 3$
 $-2x + y + 3z = 5$
 $-x + y + z = 1$
 - (iii) $x - 2y = 3$
 $3x + 2y + z = 5$
 $4y + 2z = 1$
- (b) Use the computer algebra system to determine whether each system of equations in part (a) has a unique solution and, where appropriate, solve the equations.

Exercise 25

After a social evening a club treasurer was asked how much money, in banknotes, he had deposited. He said it was £500. The members were interested in how this amount was divided between £5 notes, £10 notes and £20 notes. The treasurer said that there were three times as many £10 notes as £5 notes and there were twice as many £10 notes as £20 notes. After a short while, someone challenged him and said that could not be correct.

Follow the steps below to discover who was right.

- (a) Let the number of £5 notes be x , the number of £10 notes be y and the number of £20 notes be z , where x , y and z are integers. Write down three equations that must be satisfied if the treasurer is correct.
- (b) Express these equations in matrix form.
- (c) Use the computer algebra system to find the determinant of the coefficient matrix. What conclusion can you now draw?
- (d) Use the computer algebra system to solve the equations. Who was right: the treasurer or the challenger?

Solutions to exercises

Solution to Exercise 1

- (a) The matrix **A** has two rows and two columns, so has size 2×2 .

Similarly, matrix **B** has size 3×2 and matrix **C** has size 1×3 .

- (b) The element in row 2 and column 1 of matrix **A** is 3, so $a_{21} = 3$.
Similarly, $a_{22} = 0$.

In matrix **B**, $b_{21} = x$ and $b_{32} = 2$.

In matrix **C**, $c_{13} = 8$.

The element c_{31} does not exist because matrix **C** does not have a third row.

Solution to Exercise 2

Matrices can be added only if they have the same size. Therefore the only possible matrix sums are

$$\begin{aligned}\mathbf{A} + \mathbf{C} &= \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 7 \\ 4 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 \\ 7 & -3 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\mathbf{A} + \mathbf{E} &= \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 2 \\ 0.5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 1 \\ 3.5 & 3 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\mathbf{C} + \mathbf{E} &= \begin{pmatrix} -1 & 7 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} 6 & 2 \\ 0.5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 9 \\ 4.5 & 0 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathbf{B} + \mathbf{F} &= \begin{pmatrix} 4 & 3 \\ x & -5 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 6 & -5 \\ 1 & 2 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 10 & -2 \\ x+1 & -3 \\ 5 & 6 \end{pmatrix}.\end{aligned}$$

Matrix addition is commutative, so changing the order of the matrices in each of the sums above gives the same result.

Solution to Exercise 3

$$\begin{aligned}\text{(a) (i)} \quad \mathbf{A} - \mathbf{C} &= \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 7 \\ 4 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -8 \\ -1 & 3 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{(ii)} \quad \mathbf{F} - \mathbf{B} &= \begin{pmatrix} 6 & -5 \\ 1 & 2 \\ 5 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ x & -5 \\ 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -8 \\ 1-x & 7 \\ 5 & 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{(iii)} \quad \mathbf{A} - \mathbf{E} - \mathbf{C} &= \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} 6 & 2 \\ 0.5 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 7 \\ 4 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -3 & -10 \\ -1.5 & 0 \end{pmatrix}\end{aligned}$$

- (iv) Matrix **E** has size 2×2 and matrix **F** has size 3×2 , so $\mathbf{E} - \mathbf{F}$ is not defined.

$$\begin{aligned}\text{(v)} \quad \mathbf{D} + \mathbf{D} &= \begin{pmatrix} 5 & 0 & 3 \\ -2 & y & 6 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 3 \\ -2 & y & 6 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 0 & 6 \\ -4 & 2y & 12 \end{pmatrix}\end{aligned}$$

$$\text{(b) (i)} \quad 0.5\mathbf{A} = 0.5 \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -0.5 \\ 1.5 & 0 \end{pmatrix}$$

$$\begin{aligned}\text{(ii)} \quad 2\mathbf{D} &= 2 \begin{pmatrix} 5 & 0 & 3 \\ -2 & y & 6 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 0 & 6 \\ -4 & 2y & 12 \end{pmatrix}\end{aligned}$$

$$\text{(iii)} \quad 3\mathbf{E} = 3 \begin{pmatrix} 6 & 2 \\ 0.5 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 6 \\ 1.5 & 9 \end{pmatrix}$$

- (c) The answers to (a)(v) and (b)(ii) are the same because $\mathbf{D} + \mathbf{D} = 2\mathbf{D}$.

Solution to Exercise 4

$$\text{(a)} \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & -\frac{3}{4} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 4 & 2 \\ 1 & -6 \end{pmatrix}$$

$$\begin{aligned}\text{(b)} \quad (3x \quad -5x \quad -2x) &= x(3 \quad -5 \quad -2) \\ &\text{or } -x(-3 \quad 5 \quad 2)\end{aligned}$$

$$\begin{aligned}\text{(c)} \quad \begin{pmatrix} 3.2 & 5.4 \\ -2.4 & 1.2 \end{pmatrix} &= \frac{1}{10} \begin{pmatrix} 32 & 54 \\ -24 & 12 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 16 & 27 \\ -12 & 6 \end{pmatrix}\end{aligned}$$

Solution to Exercise 5

$$(a) \mathbf{A} + 2\mathbf{D} = \begin{pmatrix} 3 & -1 & 4 \\ -5 & 6 & 2 \end{pmatrix} + 2 \begin{pmatrix} 0 & 2 & -5 \\ 3 & 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & -6 \\ 1 & 8 & 14 \end{pmatrix}$$

$$(b) 2\mathbf{B} + 3\mathbf{C} = 2 \begin{pmatrix} -4 & 7 \\ 1 & 3 \end{pmatrix} + 3 \begin{pmatrix} 5 & -2 \\ 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 8 \\ 14 & 27 \end{pmatrix}$$

$$(c) 2\mathbf{C} - 3\mathbf{B} = 2 \begin{pmatrix} 5 & -2 \\ 4 & 7 \end{pmatrix} - 3 \begin{pmatrix} -4 & 7 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -25 \\ 5 & 5 \end{pmatrix}$$

$$(d) \mathbf{D} - 2\mathbf{A} = \begin{pmatrix} 0 & 2 & -5 \\ 3 & 1 & 6 \end{pmatrix} - 2 \begin{pmatrix} 3 & -1 & 4 \\ -5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 4 & -13 \\ 13 & -11 & 2 \end{pmatrix}$$

(e) $3\mathbf{B} - \mathbf{D}$ is undefined because $\mathbf{B}$ and $\mathbf{D}$ have different sizes.

Solution to Exercise 6

(a) The matrix $\mathbf{M}$ has size 1×3 , and matrix $\mathbf{N}$ has size 3×1 .

(b) For the product $\mathbf{MN}$ the size check is

$$1 \times \boxed{3} \quad \boxed{3} \times 1.$$

The numbers in the middle are the same, so the product $\mathbf{MN}$ is defined. Also, $\mathbf{MN}$ has size 1×1 . This product is

$$\mathbf{MN} = (2 \quad 0 \quad -3) \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix}$$

$$= (2 \times 4 + 0 \times (-1) + (-3) \times 5)$$

$$= (-7).$$

For the product $\mathbf{NM}$, the size check is

$$3 \times \boxed{1} \quad \boxed{1} \times 3.$$

The numbers in the middle are the same, so the product $\mathbf{NM}$ is defined. Also, $\mathbf{NM}$ has size 3×3 .

This product is

$$\mathbf{NM} = \begin{pmatrix} 4 \\ -1 \\ 5 \end{pmatrix} (2 \quad 0 \quad -3)$$

$$= \begin{pmatrix} 4 \times 2 & 4 \times 0 & 4 \times (-3) \\ -1 \times 2 & -1 \times 0 & -1 \times (-3) \\ 5 \times 2 & 5 \times 0 & 5 \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & -12 \\ -2 & 0 & 3 \\ 10 & 0 & -15 \end{pmatrix}.$$

Solution to Exercise 7

(a) $\mathbf{P}$ has size 3×2 and $\mathbf{Q}$ has size 2×3 . The size check for the product $\mathbf{PQ}$ is

$$3 \times \boxed{2} \quad \boxed{2} \times 3.$$

The numbers in the middle are the same, so the product $\mathbf{PQ}$ is defined. It has size 3×3 .

The product is

$$\mathbf{PQ} = \begin{pmatrix} 0 & 2 \\ 4 & -1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -4 & 0 \\ 2 & 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 10 & 6 \\ 2 & -21 & -3 \\ 7 & 37 & 15 \end{pmatrix}.$$

(b) $\mathbf{S}$ has size 3×2 and $\mathbf{R}$ has size 2×2 . The size check for the product $\mathbf{SR}$ is

$$3 \times \boxed{2} \quad \boxed{2} \times 2.$$

The numbers in the middle are the same, so the product $\mathbf{SR}$ is defined and the product has size 3×2 .

The product is

$$\mathbf{SR} = \begin{pmatrix} -2 & 6 \\ 4 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 20 \\ 14 & 11 \\ 3 & 7 \end{pmatrix}.$$

(c) $\mathbf{R}$ has size 2×2 and $\mathbf{S}$ has size 3×2 . The size check for the product $\mathbf{RS}$ is

$$2 \times \boxed{2} \quad \boxed{3} \times 2.$$

The numbers in the middle are different so the product $\mathbf{RS}$ is not defined.

Notice that $\mathbf{RS}$ is different from $\mathbf{SR}$: matrix multiplication is not commutative.

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- (d) **Q** has size 2×3 and **S** has size 3×2 .

The size check for the product **QS** is

$$2 \times \boxed{3} \quad \boxed{3} \times 2.$$

The numbers in the middle are the same, so the product **QS** is defined and the product has size 2×2 .

The product is

$$\begin{aligned} \mathbf{QS} &= \begin{pmatrix} 1 & -4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ 4 & 5 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -18 & -14 \\ 13 & 43 \end{pmatrix}. \end{aligned}$$

- (e) **S** has size 3×2 , and **Q** has size 2×3 .

The size check for the product **SQ** is

$$3 \times \boxed{2} \quad \boxed{2} \times 3.$$

The numbers in the middle are the same, so the product **SQ** is defined and the product has size 3×3 .

The product is

$$\begin{aligned} \mathbf{SQ} &= \begin{pmatrix} -2 & 6 \\ 4 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -4 & 0 \\ 2 & 5 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 10 & 38 & 18 \\ 14 & 9 & 15 \\ 3 & 14 & 6 \end{pmatrix}. \end{aligned}$$

Solution to Exercise 8

$$\begin{aligned} \text{(a) (i)} \quad \mathbf{PR} &= \begin{pmatrix} 0 & 2 \\ 4 & -1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 2 & -7 \\ 7 & 18 \end{pmatrix} \end{aligned}$$

- (ii) We have

$$2\mathbf{P} = 2 \begin{pmatrix} 0 & 2 \\ 4 & -1 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 8 & -2 \\ -6 & 10 \end{pmatrix},$$

and

$$3\mathbf{R} = 3 \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 6 & 9 \end{pmatrix},$$

so

$$\begin{aligned} 2\mathbf{P}(3\mathbf{R}) &= \begin{pmatrix} 0 & 4 \\ 8 & -2 \\ -6 & 10 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ 6 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 24 & 36 \\ 12 & -42 \\ 42 & 108 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad 6(\mathbf{PR}) &= 6 \begin{pmatrix} 4 & 6 \\ 2 & -7 \\ 7 & 18 \end{pmatrix} \\ &= \begin{pmatrix} 24 & 36 \\ 12 & -42 \\ 42 & 108 \end{pmatrix} \end{aligned}$$

- (iv) From part (ii) we have

$$\begin{aligned} -2\mathbf{P} &= - \begin{pmatrix} 0 & 4 \\ 8 & -2 \\ -6 & 10 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -4 \\ -8 & 2 \\ 6 & -10 \end{pmatrix}, \end{aligned}$$

and

$$3\mathbf{R} = \begin{pmatrix} 3 & -3 \\ 6 & 9 \end{pmatrix},$$

so

$$\begin{aligned} (-2\mathbf{P})(3\mathbf{R}) &= \begin{pmatrix} 0 & -4 \\ -8 & 2 \\ 6 & -10 \end{pmatrix} \begin{pmatrix} 3 & -3 \\ 6 & 9 \end{pmatrix} \\ &= \begin{pmatrix} -24 & -36 \\ -12 & 42 \\ -42 & -108 \end{pmatrix}. \end{aligned}$$

- (b) We see that $2\mathbf{P}(3\mathbf{R}) = 6\mathbf{PR}$,
and $(-2\mathbf{P})(3\mathbf{R}) = -6\mathbf{PR}$.

Solution to Exercise 9

- (a) We have

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 8 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 \\ 9 & -7 \end{pmatrix}, \end{aligned}$$

so

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \begin{pmatrix} 2 & 5 \\ 9 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 37 \\ 19 \end{pmatrix}.$$

Also,

$$\mathbf{AC} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix},$$

$$\mathbf{BC} = \begin{pmatrix} 4 & 2 \\ 8 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 34 \\ 13 \end{pmatrix},$$

so

$$\mathbf{AC} + \mathbf{BC} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} + \begin{pmatrix} 34 \\ 13 \end{pmatrix} = \begin{pmatrix} 37 \\ 19 \end{pmatrix}.$$

Thus $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$, as required.

(b) We have

$$\begin{aligned}\mathbf{A} + \mathbf{B} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ &= \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}(\mathbf{A} + \mathbf{B})\mathbf{C} &= \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} (a+p)x + (b+q)y \\ (c+r)x + (d+s)y \end{pmatrix} \\ &= \begin{pmatrix} ax + px + by + qy \\ cx + rx + dy + sy \end{pmatrix} \\ &= \begin{pmatrix} ax + by + px + qy \\ cx + dy + rx + sy \end{pmatrix}.\end{aligned}$$

Also,

$$\begin{aligned}\mathbf{AC} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}, \\ \mathbf{BC} &= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} px + qy \\ rx + sy \end{pmatrix},\end{aligned}$$

so

$$\begin{aligned}\mathbf{AC} + \mathbf{BC} &= \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} + \begin{pmatrix} px + qy \\ rx + sy \end{pmatrix} \\ &= \begin{pmatrix} ax + by + px + qy \\ cx + dy + rx + sy \end{pmatrix}.\end{aligned}$$

Thus $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$, as required.

Solution to Exercise 10

(a) Only square matrices (those with the same number of rows as columns) can be squared.

A has size 2×2 so can be squared.

B has size 1×1 so can be squared.

C has size 2×2 so can be squared.

D has size 2×3 so cannot be squared.

E has size 3×2 so cannot be squared.

F has size 3×3 so can be squared.

$$\begin{aligned}(b) \quad \mathbf{A}^2 &= \begin{pmatrix} -2 & -1 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 \\ 10 & 11 \end{pmatrix}\end{aligned}$$

$$\mathbf{B}^2 = (-4) \times (-4) = (16)$$

$$\mathbf{C}^2 = \begin{pmatrix} -1 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} -1 & 7 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 29 & 28 \\ 16 & 53 \end{pmatrix}$$

$$\begin{aligned}\mathbf{F}^2 &= \begin{pmatrix} 1 & -2 & 6 \\ 3 & 0 & 4 \\ 2 & 5 & 8 \end{pmatrix} \begin{pmatrix} 1 & -2 & 6 \\ 3 & 0 & 4 \\ 2 & 5 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 28 & 46 \\ 11 & 14 & 50 \\ 33 & 36 & 96 \end{pmatrix}\end{aligned}$$

Solution to Exercise 11

$$\begin{aligned}(a) \quad (i) \quad \mathbf{M}^2 &= \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 3 \\ 1 & 4 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}(ii) \quad \mathbf{M}^3 &= \mathbf{M}^2\mathbf{M} = \begin{pmatrix} 7 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 18 \\ 6 & -1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}(iii) \quad \mathbf{M}^4 &= \mathbf{M}^3\mathbf{M} = \begin{pmatrix} 17 & 18 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 52 & 33 \\ 11 & 19 \end{pmatrix}\end{aligned}$$

(b) We can use $\mathbf{M}^6 = \mathbf{M}^3\mathbf{M}^3$ or $\mathbf{M}^6 = \mathbf{M}^2\mathbf{M}^4$ or $\mathbf{M}^6 = \mathbf{M}^4\mathbf{M}^2$. For example

$$\begin{aligned}\mathbf{M}^6 &= \mathbf{M}^3\mathbf{M}^3 = \begin{pmatrix} 17 & 18 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 17 & 18 \\ 6 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 397 & 288 \\ 96 & 109 \end{pmatrix}.\end{aligned}$$

Solution to Exercise 12

(a) The label on the arc from node *A* to node *V* is 0.1. This is the proportion of water input at node *A* that flows to node *V*. So an input of 5 litres at node *A* produces an output of $0.1 \times 5 = 0.5$ litres at node *V*.

(b) Inputting 2 litres of water at node *A* gives 0.9×2 litres of water output at node *U* and 0.1×2 litres of water output at node *V*.

Inputting 1 litre of water at node *B* gives 0.5 litres of water output at node *U* and 0.5 litres of water output at node *V*.

Inputting 2 litres of water at node *C* gives 0.25×2 litres of water output at node *U* and 0.75×2 litres of water output at node *V*.

The total output at node U is thus

$$(0.9 \times 2) + 0.5 + (0.25 \times 2) = 2.8 \text{ litres.}$$

Similarly, the total output at node V is

$$(0.1 \times 2) + 0.5 + (0.75 \times 2) = 2.2 \text{ litres.}$$

- (c) The total outputs are found using a similar method to that of part (b).

Therefore the total output at U is

$$0.9x + 0.5y + 0.25z \text{ litres,}$$

and the total output at V is

$$0.1x + 0.5y + 0.75z \text{ litres.}$$

- (d) The input vector is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

- (e) The matrix representing the network has three columns (one for each input node) and two rows (one for each output node). The element in the first row and first column is the label on the arc from the first input node (A) to the first output node (U), which is 0.9. The element in the first row and second column is the label on the arc from the second input node (B) to the first output node (U), which is 0.5. Continuing in this way gives the following matrix:

$$\begin{array}{ccc} & A & B & C \\ U & \begin{pmatrix} 0.9 & 0.5 & 0.25 \end{pmatrix} \\ V & \begin{pmatrix} 0.1 & 0.5 & 0.75 \end{pmatrix} \end{array}$$

Solution to Exercise 13

- (a) The label on each arc starting from node A represents the proportion of water that flows through the corresponding pipe when water is poured in at A . These proportions have sum 1. The proportion of this water flowing from A to U is 0.3 and the proportion of this water flowing from A to W is 0.2. Thus the missing number for the arc from A to V is $1 - (0.3 + 0.2) = 0.5$.

Similarly, the missing number for the arc from B to U is $1 - (0.15 + 0.5) = 0.35$.

- (b) The matrix is

$$\begin{array}{ccc} & A & B \\ U & \begin{pmatrix} 0.3 & 0.35 \end{pmatrix} \\ V & \begin{pmatrix} 0.5 & 0.15 \end{pmatrix} \\ W & \begin{pmatrix} 0.2 & 0.5 \end{pmatrix} \end{array}$$

- (c) The input vector is $\begin{pmatrix} x \\ y \end{pmatrix}$, and the output vector is $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$.

The relationship between these two vectors is represented by the matrix equation

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0.3 & 0.35 \\ 0.5 & 0.15 \\ 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution to Exercise 14

- (a) The matrix that represents the network with nodes A, B, C, K, L and M is

$$\begin{array}{ccc} & A & B & C \\ K & \begin{pmatrix} 0.5 & 0.2 & 0.1 \end{pmatrix} \\ L & \begin{pmatrix} 0.5 & 0.8 & 0.3 \end{pmatrix} \\ M & \begin{pmatrix} 0 & 0 & 0.6 \end{pmatrix} \end{array}$$

The matrix equation giving the output

vector $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ for the input vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is

$$\begin{pmatrix} k \\ l \\ m \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.5 & 0.8 & 0.3 \\ 0 & 0 & 0.6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

- (b) The matrix that represents the network with nodes K, L, M, G and H is

$$\begin{array}{ccc} & K & L & M \\ G & \begin{pmatrix} 0.3 & 0.1 & 0.5 \end{pmatrix} \\ H & \begin{pmatrix} 0.7 & 0.9 & 0.5 \end{pmatrix} \end{array}$$

The matrix equation giving the output

vector $\begin{pmatrix} g \\ h \end{pmatrix}$ for the input vector $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ is

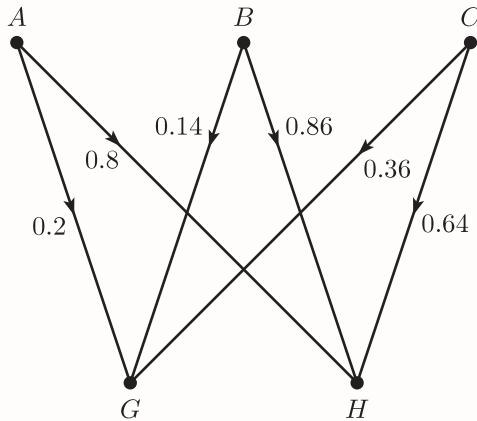
$$\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} 0.3 & 0.1 & 0.5 \\ 0.7 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}.$$

- (c) The matrix equation giving the output vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for an input vector $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ in the combined network is obtained using parts (a) and (b) as follows:

$$\begin{aligned}
\begin{pmatrix} g \\ h \end{pmatrix} &= \begin{pmatrix} 0.3 & 0.1 & 0.5 \\ 0.7 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix} \\
&= \begin{pmatrix} 0.3 & 0.1 & 0.5 \\ 0.7 & 0.9 & 0.5 \end{pmatrix} \left(\begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.5 & 0.8 & 0.3 \\ 0 & 0 & 0.6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} 0.3 & 0.1 & 0.5 \\ 0.7 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.5 & 0.8 & 0.3 \\ 0 & 0 & 0.6 \end{pmatrix} \right) \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= \begin{pmatrix} 0.2 & 0.14 & 0.36 \\ 0.8 & 0.86 & 0.64 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.
\end{aligned}$$

- (d) A simplified network equivalent to the original one is given below. The labels can be read from the product matrix that represents the combined network, that is

$$\begin{array}{ccc}
& A & B & C \\
\begin{array}{l} G \\ H \end{array} & \begin{pmatrix} 0.2 & 0.14 & 0.36 \\ 0.8 & 0.86 & 0.64 \end{pmatrix}.
\end{array}$$



- (e) Inputs of 120, 50 and 100, at nodes A , B and C respectively, are represented by

$$\text{the input vector } \begin{pmatrix} 120 \\ 50 \\ 100 \end{pmatrix}.$$

Using the equation for the combined network from part (c) gives the corresponding output vector

$$\begin{aligned}
\begin{pmatrix} g \\ h \end{pmatrix} &= \begin{pmatrix} 0.2 & 0.14 & 0.36 \\ 0.8 & 0.86 & 0.64 \end{pmatrix} \begin{pmatrix} 120 \\ 50 \\ 100 \end{pmatrix} \\
&= \begin{pmatrix} 67 \\ 203 \end{pmatrix}.
\end{aligned}$$

So 67 people arrive at airport G and 203 arrive at airport H .

Solution to Exercise 15

- (a) The following matrix products can be formed: **PI**, **IP**, **QI**, **IR**.

For the product **IQ** the size check is

$$2 \times \boxed{2} \quad \boxed{3} \times 2.$$

The numbers in the middle are not the same, so the product **IQ** is not defined.

For the product **RI** the size check is

$$2 \times \boxed{3} \quad \boxed{2} \times 2.$$

The numbers in the middle are not the same, so the product **RI** is not defined.

$$(b) \quad \mathbf{PI} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} = \mathbf{P}$$

$$\mathbf{IP} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix} = \mathbf{P}$$

$$\mathbf{QI} = \begin{pmatrix} 6 & -1 \\ -4 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ -4 & 2 \\ 0 & 3 \end{pmatrix} = \mathbf{Q}$$

$$\mathbf{IR} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 2 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 \\ 2 & 1 & 5 \end{pmatrix} = \mathbf{R}$$

Solution to Exercise 16

- (a) The following matrix products can be formed: **LI**, **IL**, **IM**, **NI**.

For the product **MI** the size check is

$$3 \times \boxed{2} \quad \boxed{3} \times 3.$$

The numbers in the middle are not the same, so the product **MI** is not defined.

For the product **IN** the size check is

$$3 \times \boxed{3} \quad \boxed{2} \times 3.$$

The numbers in the middle are not the same, so the product **IN** is not defined.

$$(b) \quad \mathbf{LI} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 3 & -5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 3 & -5 \end{pmatrix} = \mathbf{IL} = \mathbf{L}$$

$$\mathbf{IM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 3 & 5 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \\ 3 & 5 \\ -1 & 0 \end{pmatrix} = \mathbf{M}$$

$$\begin{aligned}\mathbf{N}\mathbf{I} &= \begin{pmatrix} 0 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix} = \mathbf{N}\end{aligned}$$

Solution to Exercise 17

Multiplying a matrix by its inverse gives an identity matrix, that is, a matrix that has ones down the leading diagonal and zeros elsewhere. This is the case in each of the products below, so in each case $\mathbf{Q}$ is the inverse of $\mathbf{P}$.

$$\begin{aligned}\text{(a) } \mathbf{PQ} &= \begin{pmatrix} -6 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{QP} &= \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -6 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{(b) } \mathbf{PQ} &= \begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & -4 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} -19 & 10 & 2 \\ 2 & -1 & 0 \\ -10 & 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{QP} &= \begin{pmatrix} -19 & 10 & 2 \\ 2 & -1 & 0 \\ -10 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 2 & -1 & -4 \\ 0 & 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

Solution to Exercise 18

(a) Since

$$\det \mathbf{A} = 7 \times 2 - 2 \times 6 = 14 - 12 = 2,$$

we have

$$\mathbf{A}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -2 \\ -6 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & \frac{7}{2} \end{pmatrix}.$$

(b) Since

$$\begin{aligned}\det \mathbf{B} &= 3 \times 2 - (-6) \times (-1) \\ &= 6 - 6 = 0,\end{aligned}$$

the matrix $\mathbf{B}$ is not invertible; that is, $\mathbf{B}^{-1}$ does not exist.

(c) Since

$$\begin{aligned}\det \mathbf{C} &= \frac{1}{3} \times \left(-\frac{1}{2}\right) - \frac{1}{5} \times 5 \\ &= -\frac{1}{6} - 1 = -\frac{7}{6},\end{aligned}$$

we have

$$\mathbf{C}^{-1} = \frac{1}{-\frac{7}{6}} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{5} \\ -5 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & \frac{6}{35} \\ \frac{30}{7} & -\frac{2}{7} \end{pmatrix}.$$

Solution to Exercise 19

(a) The determinant of $\mathbf{A}$ is

$$a \times a - b \times (-b) = a^2 + b^2.$$

(b) The matrix $\mathbf{A}$ has an inverse provided that its determinant is non-zero. As a^2 and b^2 are always greater than or equal to zero, the determinant is zero only when $a = b = 0$. Thus $\mathbf{A}$ has an inverse for all values of a and b except $a = b = 0$, and the inverse is

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{a^2 + b^2} & \frac{-b}{a^2 + b^2} \\ \frac{b}{a^2 + b^2} & \frac{a}{a^2 + b^2} \end{pmatrix}.\end{aligned}$$

Solution to Exercise 20

(a) The determinant of $\mathbf{A}$ is $6 - (-1) = 7$.

$$\begin{aligned}\text{(b) } \mathbf{AB} &= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}, \\ &\text{and}\end{aligned}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}.$$

Both answers are the same and can be rewritten as $7 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So if each term in $\mathbf{B}$ is divided by 7, the determinant of $\mathbf{A}$, then the resulting matrix will be the inverse of $\mathbf{A}$. That is, $\frac{1}{7}\mathbf{B} = \mathbf{A}^{-1}$.

Similarly, $\frac{1}{7}\mathbf{A} = \mathbf{B}^{-1}$.

Solution to Exercise 21

Using the computer algebra system we find that the determinant of $\mathbf{A}$ is -10 , so the matrix is invertible.

We have

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{5} & \frac{1}{2} & -\frac{3}{10} \\ \frac{2}{5} & 0 & \frac{1}{5} \\ -\frac{4}{5} & \frac{1}{2} & \frac{1}{10} \end{pmatrix}.$$

The determinant of $\mathbf{B}$ is 0, so it is non-invertible.

The determinant of $\mathbf{C}$ is 39, so the matrix is invertible, and we find that

$$\mathbf{C}^{-1} = \begin{pmatrix} \frac{14}{39} & -\frac{4}{39} & \frac{25}{39} & \frac{38}{39} \\ \frac{5}{39} & -\frac{7}{39} & -\frac{5}{39} & \frac{8}{39} \\ \frac{11}{39} & \frac{8}{39} & -\frac{11}{39} & \frac{2}{39} \\ \frac{2}{39} & \frac{5}{39} & -\frac{2}{39} & \frac{11}{39} \end{pmatrix}.$$

Solution to Exercise 22

- (a) The matrix form of the equations is

$$\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

The coefficient matrix is $\begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$, which has determinant

$$4 \times 1 - 1 \times 3 = 1.$$

The inverse of the coefficient matrix is therefore

$$\frac{1}{1} \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \end{pmatrix}.$$

The solution is $x = -3$, $y = 14$.

- (b) The matrix form of the equations is

$$\begin{pmatrix} -2 & 5 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}.$$

The coefficient matrix is $\begin{pmatrix} -2 & 5 \\ 3 & 7 \end{pmatrix}$, which has determinant

$$-2 \times 7 - 5 \times 3 = -29.$$

The inverse of the coefficient matrix is therefore

$$\frac{1}{-29} \begin{pmatrix} 7 & -5 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{7}{29} & \frac{5}{29} \\ \frac{3}{29} & \frac{2}{29} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{7}{29} & \frac{5}{29} \\ \frac{3}{29} & \frac{2}{29} \end{pmatrix} \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{66}{29} \\ \frac{20}{29} \end{pmatrix}.$$

The solution is $x = -\frac{66}{29}$, $y = \frac{20}{29}$.

- (c) The matrix form of the equations is

$$\begin{pmatrix} \frac{1}{3} & 3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}.$$

The coefficient matrix is $\begin{pmatrix} \frac{1}{3} & 3 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$, which has determinant

$$\frac{1}{3} \times \left(-\frac{3}{2}\right) - 3 \times \frac{1}{2} = -2.$$

The inverse of the coefficient matrix is therefore

$$\frac{1}{-2} \begin{pmatrix} -\frac{3}{2} & -3 \\ -\frac{1}{2} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{3}{2} \\ \frac{1}{4} & -\frac{1}{6} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{3}{2} \\ \frac{1}{4} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} -8 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}.$$

The solution is $x = 3$, $y = -3$.

Solution to Exercise 23

- (a) The matrix form of the equations is

$$\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$7 \times 1 - 3 \times (-3) = 16.$$

This is non-zero, so the coefficient matrix is invertible and the equations have a unique solution.

- (b) First we rearrange each equation so that the unknowns are on the left-hand side in the same order, and the constant terms are on the right-hand side:

$$-4x + 6y = -6$$

$$6x - 9y = 5.$$

The matrix form of the equations is

$$\begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$(-4) \times (-9) - 6 \times 6 = 0.$$

This is zero, so the coefficient matrix is non-invertible and the equations do not have a unique solution.

Solution to Exercise 24

- (a) The equations can be written in matrix form as given below.

$$(i) \quad \begin{pmatrix} 3 & 2 & 1 \\ -5 & 2 & 2 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \\ 1 \end{pmatrix}$$

$$(ii) \quad \begin{pmatrix} 3 & -3 & 2 \\ -2 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

$$(iii) \quad \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

- (b) Using the computer algebra system we find that the determinants of the coefficient matrices are, respectively, 85, -5 and 12. Each determinant is non-zero, so each system of equations has a unique solution as found below.

For the system of equations in part (i) we have

$$\begin{pmatrix} 3 & 2 & 1 \\ -5 & 2 & 2 \\ 2 & -3 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{12}{85} & -\frac{9}{85} & \frac{2}{85} \\ \frac{19}{85} & \frac{7}{85} & -\frac{11}{85} \\ \frac{11}{85} & \frac{13}{85} & \frac{16}{85} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{12}{85} & -\frac{9}{85} & \frac{2}{85} \\ \frac{19}{85} & \frac{7}{85} & -\frac{11}{85} \\ \frac{11}{85} & \frac{13}{85} & \frac{16}{85} \end{pmatrix} \begin{pmatrix} 4 \\ 15 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

The solution is $x = -1$, $y = 2$, $z = 3$.

For the system of equations in part (ii) we have

$$\begin{pmatrix} 3 & -3 & 2 \\ -2 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{5} & -1 & \frac{11}{5} \\ \frac{1}{5} & -1 & \frac{13}{5} \\ \frac{1}{5} & 0 & \frac{3}{5} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -1 & \frac{11}{5} \\ \frac{1}{5} & -1 & \frac{13}{5} \\ \frac{1}{5} & 0 & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{8}{5} \\ -\frac{9}{5} \\ \frac{6}{5} \end{pmatrix}.$$

The solution is $x = -\frac{8}{5}$, $y = -\frac{9}{5}$, $z = \frac{6}{5}$.

For the system of equations in part (iii) we have

$$\begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{6} & -\frac{1}{12} \\ 1 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{2} & \frac{1}{6} & -\frac{1}{12} \\ 1 & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{4} \\ 2 \end{pmatrix}.$$

The solution is $x = \frac{3}{2}$, $y = -\frac{3}{4}$, $z = 2$.

Solution to Exercise 25

- (a) If x, y and z represent the numbers of £5 notes, £10 notes and £20 notes respectively, then the total sum of money is given by

$$5x + 10y + 20z = 500.$$

There were said to be three times as many £10 notes as £5 notes, so $y = 3x$ or $3x - y = 0$.

There were said to be twice as many £10 notes as £20 notes, so $y = 2z$ or $y - 2z = 0$.

- (b) In matrix form this gives

$$\begin{pmatrix} 5 & 10 & 20 \\ 3 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 500 \\ 0 \\ 0 \end{pmatrix}$$

- (c) The determinant of the coefficient matrix is 130, so the three equations have a unique solution.

(If you expressed the equations in part (a) in a different form from those given above, your answer to part (b) and your answer for the determinant in part (c) may be different from the answers given here.)

- (d) We have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 10 & 20 \\ 3 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 500 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{100}{13} \\ \frac{300}{13} \\ \frac{150}{13} \end{pmatrix}.$$

The solution is $x = \frac{100}{13}$, $y = \frac{300}{13}$, $z = \frac{150}{13}$.

Although the three equations can be solved simultaneously, they do not produce integer answers, so the challenger was right.